Heat flowing from cold to hot without external intervention by using a "thermal inductor"

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The cooling of a quantity of boiling water all the way down to freezing by simply thermally connecting it to a thermal bath held at ambient temperature without external intervention would be quite unexpected. We describe the equivalent of a "thermal inductor" composed of a Peltier element and an electric inductance, which can drive the temperature difference between two bodies to change sign by imposing a certain inertia on the flow of heat, thereby enabling continuing heat transfer from the chilling body to its warmer counterpart. We show theoretically and experimentally that such a process is possible and fully complies with the second law of thermodynamics. With further progress in thermoelectric materials, it could serve to cool hot materials well below ambient temperature without external energy supply.

According to a popular perception of the second law of thermodynamics, a hot body with temperature T_b that is connected to a colder body at temperature T_r , approaches thermodynamic equilibrium with strictly $T_b > T_r$, and T_b is expected to monotonically fall as a function of time t because heat is not supposed to flow by itself from a cold to a warmer body (Figs. 1A and 1C). Any undershooting or oscillatory behavior of T_b with respect to T_r (Figs. 1B and 1D) with a reversing direction of the heat flow and transferring heat from cold to hot, would normally be ascribed to an active intervention to remove heat with external work to be done. We describe a simple thermal connection, acting without any external or internally hidden source of power, which drives the temperature difference between an initially warm and a colder body to change sign by imposing a certain thermal inertia on the flow of heat. We demonstrate in an experiment that such a process can occur through a series of quasi-equilibrium states, and show that it still fully complies with the second law of thermodynamics in the sense that the entropy of the whole system monotonically increases over time, albeit heat is temporarily flowing from cold to hot.

This thermal connection consists of an ideal electrical inductor with inductance L and a Peltier element with Peltier coefficient $\Pi = \alpha T$, forming a closed electrical circuit [1] (Fig. 2).

Here, α stands for the combined Seebeck coefficient of the used thermoelectric materials, and T is the absolute temperature of the considered junction between these materials. When an electric current I is flowing through, heat Q is generated or absorbed at a rate $\dot{Q} = \Pi I = \alpha T I$, respectively, depending on the direction of the current. A body with heat capacity C and a thermal reservoir (or two finite bodies) are each in thermal contact with the opposite sides of the Peltier element providing a thermal link by its thermal conductance k. The process is described by Kirchhoff's voltage law in Eq. 1a containing the generated thermoelectric voltage $\alpha(T_b - T_r)$, and by the thermal balance equations Eqs. 1b and 1c for the rates of heat removed from (or added to) one body ($\dot{Q}_b$) and to (from) the other body or the thermal reservoir ($\dot{Q}_r$), respectively,

$$L\dot{I} + RI = \alpha(T_b - T_r), \tag{1a}$$

$$\dot{Q}_b = -\alpha T_b I + \tfrac{1}{2} RI^2 - k(T_b - T_r), \tag{1b}$$

$$\dot{Q}_r = +\alpha T_r I + \tfrac{1}{2} RI^2 + k(T_b - T_r), \tag{1c}$$

where R is the internal resistance of the Peltier element. We temporarily ignore parasitic effects due to other sources of electrical resistance or thermal transport through leads or convection, and the heat-capacity contribution of the Peltier element is thought to be absorbed in C. We count $\dot{Q} > 0$ for a heat input; the choices of the signs of I and α in the Eqs. 1 turn out to be unimportant, however. The dissipated Joule heating power RI^2 is regarded to be equally distributed to both sides of the Peltier element. The set of equations Eqs. 1, but without the inductive term $L\dot{I}$, is standard to describe the flow of heat and charge in a Peltier element [2,3].

To consider the situation for the actual experiment to be presented further below, where a finite heat capacity C is connected to an infinite thermal reservoir as shown in Figs. 1A and 1B we combine $\dot{Q}_b = C\dot{T}_b$ and $T_r = const.$ with the Eqs. 1a and 1b and obtain a nonlinear differential equation for $I(t)$, namely

$$LC\ddot{I} + (RC + kL)\dot{I} + (kR + \alpha^2 T_r)I + \tfrac{1}{2}\alpha RI^2 + \alpha LI\dot{I} = 0\,. \tag{2}$$

This equation can be easily numerically solved. The time evolution $T_b(t)$ could then be obtained by inserting the corresponding solution $I(t)$ into Eq. 1a. For a systematic analysis we restrict ourselves to $\Delta_0 = T_b(0) - T_r << T_r$ with temperature independent α, k, R and C, valid within a sufficiently narrow temperature interval $\pm\Delta_0$ around T_r. Then, the last two terms of the differential equation Eq. 2 are negligible because $\tfrac{1}{2}RI^2 + LI\dot{I} < RI^2 + LI\dot{I} = \alpha(T_b - T_r)I$ $<< \alpha T_r I$, and we end up with the equation for a damped harmonic oscillator,

$$LC\ddot{I} + (RC + kL)\dot{I} + (kR + \alpha^2 T_r)I \approx 0\,. \tag{3}$$

The initial conditions for $t = 0$, i.e., the time when the virgin thermal connection is inserted, are $I(0) = 0$ and $T_b(0) - T_r = \Delta_0$, thereby fixing $\dot{I}(0) = \alpha\Delta_0 / L$. The corresponding solution may show either a damped or an oscillatory behavior, respectively, depending on the combination of the constant factors in the equation. To achieve a possible undershooting of $T_b(t)$ below T_r, we focus only on oscillatory solutions of $I(t)$. The condition for their occurrence, $4\alpha^2 T_r > LC(k/C - R/L)^2$, can always be fulfilled for any value of α, if L is chosen as $L^* = RC/k$. While R and k are given by the characteristics of the Peltier element, C is limited only by the heat capacity of the Peltier element but can otherwise be chosen at will. The solution of interest is $I(t) = I_0 \exp(-t/\tau)\sin(\omega t)$, with τ and ω matched to fulfill the differential equation, and $I_0 = \alpha\Delta_0 / (L\omega)$. The corresponding phase-shifted solution for the temperature is $T_b(t) - T_r = \Delta_0 \exp(-t/\tau)\cos(\omega t - \delta)/\cos\delta$ with $\tan\delta = (R - L/\tau)/L\omega$. We now seek the particular solution realizing the weakest possible damping of $I(t)$ within an oscillation cycle. This occurs for a maximum value of $\omega\tau$ where $L = L^*$, $\omega = \omega^* = \sqrt{\alpha^2 T_r k / (RC^2)}$ and $\tau = \tau^* = C/k$. Introducing the standard definition of the dimensionless figure of merit $Z_T = \alpha^2 T_r / kR = (\omega^* \tau^*)^2$ of a Peltier element [4] the first minimum of $T_b(t)$ is attained with these parameters at $t_{\min} = \beta\tau^*$ with $\beta = (\pi/2 + \arctan\sqrt{Z_T})/\sqrt{Z_T}$ for

$$T_{b,\min} = T_r - \Delta_0 \exp(-\beta)\sqrt{Z_T/(Z_T+1)} < T_r . \tag{4}$$

If expressed by the dimensionless quantities $(T_b(t) - T_r)/\Delta_0$ and t/τ^*, the time evolution and $(T_{b,\min} - T_r)/\Delta_0$, a measure for the maximum obtained cooling effect, only depend on Z_T but are independent of the thermal load C and the other parameters of the system. In Fig. 3A we show a series of corresponding numerically obtained $T_b(t)$ curves for different values of Z_T, expressed in these dimensionless units, with $T_b(0) - T_r = \Delta_0 = 80$ K and $T_r = 293$ K to mimic a realistic scenario.

According to Eq. 4, the minimum temperature decreases with increasing Z_T, but is limited to $T_{b,\min} = T_r - \Delta_0$ for $Z_T \to \infty$ so that no catastrophic oscillation can occur. In that limit, the temperature of the body would oscillate between the extremal values $T_r + \Delta_0$ and $T_r - \Delta_0$. If the thermal connection were removed after the body has reached its minimum temperature, T_b would stay at $T_{b,\min} < T_r$ under perfectly isolated conditions as sketched in Fig. 1B. Removing it in a state where $I = 0$ even leaves the thermal connection in its original, virgin state at $T_b = T_r - \Delta_0 \exp(-\pi/\sqrt{Z_T})$ but still with $T_b < T_r$ (inset of Fig. 3A). Any external work associated with the act of removing the thermal connection could be made infinitesimally small, e.g., by opening a nanometer-sized gap between the body and the thermal connection. If the connection is not removed at all, $T_b(t)$ exhibits a damped oscillation around T_r, approaching thermal equilibrium with eventually $T_b = T_r$. We note that the maximum possible cooling effect is not reached for the above parameters, but for $L_{opt} = \lambda\, L^*$ with $\lambda(Z_T) > 1$ (see section S1, Supplementary Materials). The corresponding solutions for $I(t)$ are over-damped for $Z_T < 1/3$, but the temperature of the body still undershoots T_r for all values of $Z_T > 0$.

In a closely related scenario, two finite bodies with identical heat capacities $2C$ and different initial temperatures $T_b(0)$ and $T_r(0)$ are thermally connected in the same way, and observed under completely isolated conditions (Fig. 1D). In the limit $\Delta_0 = T_b(0) - T_r(0) << \bar{T}$ with the mean initial temperature $\bar{T} = [T_b(0) + T_r(0)]/2$, we end up with the same simplified differential equation Eq. 3 for $I(t)$ but with T_r replaced by $\bar{T}$ (see section S2, Supplementary Materials). In Fig. 3B, we show the resulting counter-oscillating temperatures $T_b(t)$ and $T_r(t)$ for $Z_T = 5$, together with the average temperature $T_{av} = [T_b(t) + T_r(t)]/2 \le \bar{T}$, which is not constant but reaches local minima around the times when the two temperatures are equal.

Each time when $T_b - T_r$ changes its sign (this occurs for the first time as soon as T_b drops below T_r, for $L = L^*$ after $t_0 = \pi / 2\omega^*$, Figs. 1 and 3), heat is still continuously flowing from the cold to the warmer object (dark/purple arrows in Fig. 1) until $|T_b - T_r|$ reaches its maximum, where the direction of the heat flow is reversed. Virtually all of this heat is driven directly away by the moving charge carriers from the cold to the warm end, without being temporarily stored as energy of the magnetic field residing in the inductor. The maximum amount of magnetic energy, $\frac{1}{2} L I^2 < \frac{1}{2} L I_0^2 = \alpha^2 \Delta_0^2 / 2 L^* \omega^{*2}$ is less than a fraction $\Delta_0 / T_r << 1$ of the excess heat $\sim C\Delta_0$ that has been initially stored in the warmer body. In this sense, the electrical inductor acts, in interplay with the Peltier element, only as the driver of the temperature oscillation by imposing a certain thermal inertia that temporarily counteracts the flow of heat dictated by Fourier's law. Thus, we can interpret the role of the circuit as that of a "thermal inductor". In full analogy to the self-inductance L of an electrical inductor generating a voltage difference ΔV according to $L\dot{I} = -\Delta V$, we can even ascribe to the present circuit a thermal self-inductance $L_{th} = L / (\alpha^2 T_r)$ [1], obeying $L_{th} \dot{I}_{th} = L_{th} \ddot{Q} = -\Delta T$, see section S3, Supplementary Materials.

The order of magnitude of the rate of heat flowing between the objects can be chosen arbitrarily low by adjusting the thermal load C, which guarantees that the processes, albeit irreversible, can run in a quasistatic way and pass through a series of quasi-equilibrium states with well defined thermodynamic potentials and state variables of the bodies. This is in marked contrast to non-equilibrium oscillatory processes such as the Belousov-Zhabotinski chemical reaction [5,6], other oscillations in complex systems far from thermodynamic equilibrium [7], or to "thermal-inductor" type of behaviors associated with the convection of heated fluids [8] and transient switching operations in light-emitting diodes [9].

The proof that the processes as described above do not violate the second law of thermodynamics is simple. The total entropy production is $\dot{S}_{tot} = \dot{S}_b + \dot{S}_r = \dot{Q}_b / T_b + \dot{Q}_r / T_r$. The empty current-carrying inductor does not contribute to the entropy balance since an associated magnetic contribution to the Gibb's free energy does not depend on the temperature. While the terms in the Eqs. 1b and 1c related to the Peltier element cancel out, $RI^2 > 0$,

$-(T_b - T_r)/T_b + (T_b - T_r)/T_r \geq 0$, and we have indeed $\dot{S}_{tot} > 0$. The temperatures $T_b(t)$ and $T_r(t)$ counter-oscillate in the second scenario (Figs. 1D and 3B) and repeatedly match around $\overline{T}$, which seems to be at odds with the expectation that the total entropy of the two bodies with $T_b = T_r$ must be larger than with $T_b \neq T_r$. As T_{av} is not constant but lowest for $T_b \approx T_r < \overline{T}$, the total entropy is indeed a monotonously increasing function of time (inset of Fig. 3B). The fact that heat temporarily flows from cold to hot in such processes, despite the indisputable increase of the total entropy with time, calls for a further explanation, however. Although no external intervention is generating the inversion of the temperature gradients and forcing heat to flow from cold to hot, the inductor is, as an integrated part of the thermal connection, perpetually changing its state due to the oscillatory electric current. Therefore, the described processes are also in full conformity with the original postulate of the second law of thermodynamics by Clausius, stating that a flow of heat from cold to hot must be associated with "some other change, connected therewith, occurring at the same time" [10].

In a real experiment, measurements of sizeable temperature oscillations face certain challenges. At present, the most efficient Peltier elements have a maximum $Z_T \approx 2$ [11]. In the scenario of cooling a body from 100 ºC down to 0 ºC by connecting it to a heat sink at 20 ºC, a $Z_T \approx 5.0$ would be required (Fig. 3A), which is out of reach of the present technology. A further challenge is the choice of the inductivity value *L*. It should be large enough to allow for the cooling of a sizeable amount of material while keeping *k* as small as possible to maximize Z_T. With $\tau^* = C/k$ of the order of several seconds and typical internal resistances of Peltier elements $R \approx 0.1\ \Omega$ and higher, $L_{opt} > L^* = R\tau^*$ must be of the order of 1 H or larger, although a useful cooling effect could still be achieved for $L < L^*$. Inductors with such a large inductance suffer from a considerable internal resistance, however, thereby reducing the cooling performance below the expectations.

The use of an active gyrator-type substitute of a real inductor [12] allowing for almost arbitrarily large values of *L* with negligible effective internal resistance circumvents this problem. Although the thermal connection can then no longer be considered as strictly operating "without external intervention", no net external work is performed on the system. In Fig. 4 we present the results of a series of corresponding proof-of-principle measurements according to the experimental scheme shown in Fig. 1B and analyzed in Fig. 3A. We used a commercially available Peltier element, a cube of ≈ 1 cm^3 copper as the thermal load, and a massive copper base acting as the thermal reservoir held at T_r = 295 K, for four different values of the nominal inductance *L* of the gyrator (*L* = 33.2 H, 53.6 H, 90.9 H and 150 H). Initially heated to 377 K (≈ 104 ºC), T_b dropped by up to ≈ 2.7 K below the base temperature T_r, clearly demonstrating that heat has indeed been flowing from the chilling copper cube to the thermal reservoir as soon as T_b fell below 295 K. In the same figure, we also show the results of a global fit to both the measured $T_b(t)$ and *I*(t) data according to the relations given in this work, with free parameters *C*, *R*, *k* and Z_T. We obtain an excellent agreement for an effective

Z_T = 0.432. The used values L = 33.2 H and 90.9 H are very close to the respective L^* = 34.3 H and L_{opt} = 94.4 H, respectively (see Materials and Methods).

To summarize, we have shown both theoretically and experimentally that the use of a thermal connection containing a variant of a "thermal inductor" can drive the flow of heat from a cold to a hot object without external intervention, while still fully complying with the second law of thermodynamics. Despite the conceptual simplicity of the described process and based on the laws of classical physics, it has, to our knowledge, never been considered in the literature. With future progress in materials research, the technique may become technically useful and allow for the cooling of hot materials well below the ambient temperature without the need for an external energy supply.

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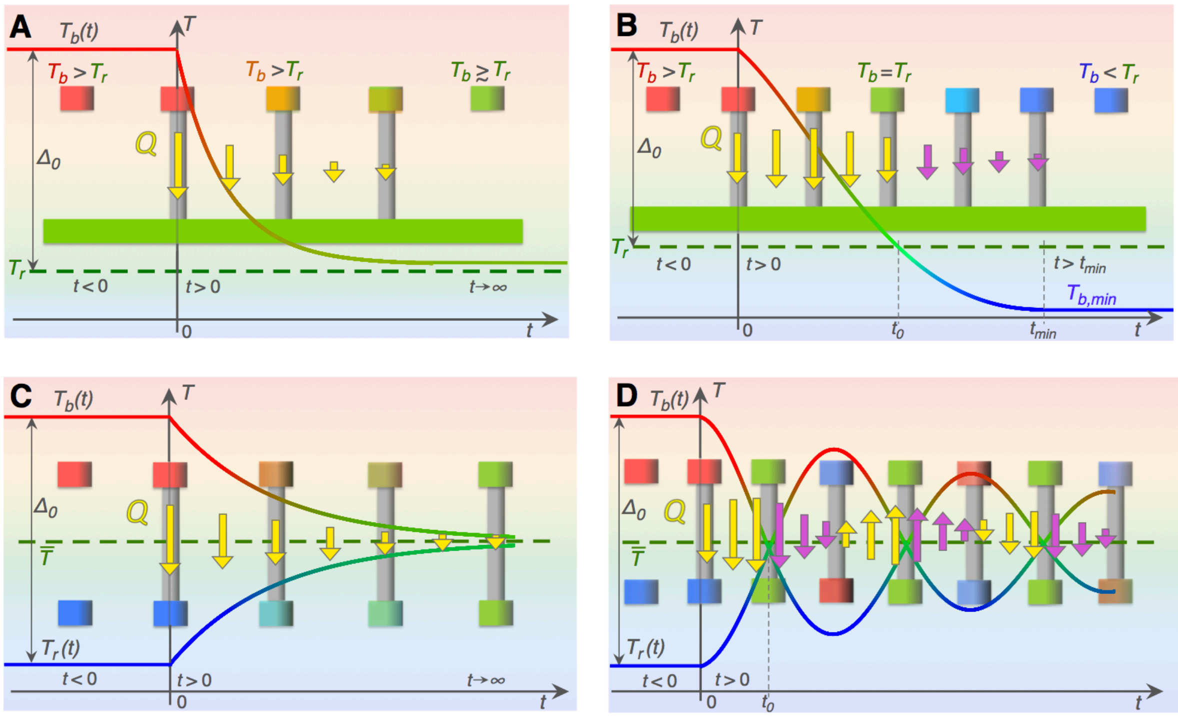

Fig. 1. Sketch of two situations in which objects with different temperatures are thermally connected. Arrows represent the direction of the flow of heat from (light/yellow) or to (dark/purple) the respective warmer object. (**A**) When an initially hot body is thermally connected at time $t = 0$ to a colder thermal reservoir held at temperature T_r, its temperature T_b is expected to drop monotonically by the loss of heat Q to the colder reservoir, and to approach T_r in the limit $t \to \infty$. (**B**) Sketch of a process in which T_b undershoots the temperature of the reservoir for $t > t_0$, and heat Q is thereafter temporarily transferred from the chilling body to the warmer reservoir. The lowest temperature of the body $T_{b,\min} < T_r$ is reached at $t = t_{\min}$ when the connection can be removed. (**C**) Two similarly connected finite heat capacities are expected to smoothly approach thermodynamic equilibrium at a mean temperature $\overline{T}$, with heat flowing in one direction only and always $T_b > T_r$. (**D**) Two bodies showing opposite oscillations in temperature, with an alternating direction of the heat flow and a repeated temporary transfer of heat from cold to hot.

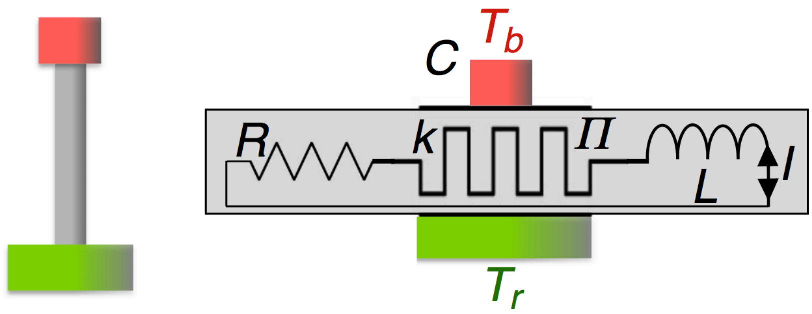

Fig. 2. Equivalent electrical network of the considered thermal connection between a body with heat capacity *C* at temperature T_b and another body or a thermal reservoir at T_r. It consists of a Peltier element (Π) with internal resistance R and thermal conductance k, in a closed circuit with an ideal inductance L. The oscillatory current I is ultimately driven by the voltages supplied by the thermoelectric effect due to the temperature difference between the cold and the hot end of the Peltier element, and the induced voltage $L\dot{I}$ across L, see Eq. 1a.

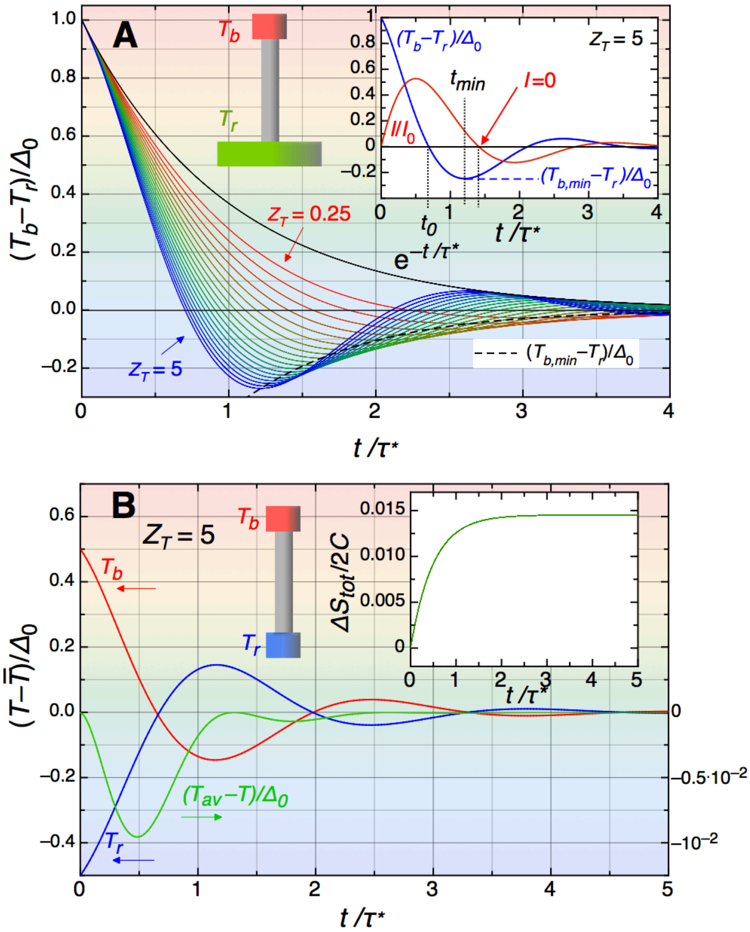

Fig. 3. Evolution of the temperature difference between a cooling body and a thermal bath or another finite body, which are connected in an experiment using a "thermal inductor".

(**A**) Normalized temperature difference $(T_b(t)-T_r)/\Delta_0$ between a finite body and a thermal reservoir for $L = L^* = RC/k$, and Z_T between 0.25 (red) and 5 (blue) in steps of 0.25. The time is in units of $\tau^* = C/k$. The black line represents a corresponding relaxation process with a time constant τ^*, which would take place if the Peltier circuit were interrupted from the beginning. If the thermal connection is not removed after reaching the respective $T_{b,\min}$ (dashed line), $T_b(t)$ approaches thermal equilibrium with eventually $T_b = T_r$ in all cases. The inset shows the damped oscillations of both $T_b(t)$ and $I(t)$.

(**B**) Temperatures $T_b(t)$ and $T_r(t)$ of two connected finite bodies with equal heat capacities, relative to the mean initial temperature $\bar{T} = [T_b(0)+T_r(0)]/2$ and normalized to the initial temperature difference Δ_0, for Z_T = 5 (time in units of τ^*). T_{av} denotes their averaged value with respect to $\bar{T}$, showing local minima around $T_b \approx T_r$ (the numbers for T_{av} were calculated for $\Delta_0 / T_r = 0.27$). The inset shows the evolution of the total entropy gain as a function of time in corresponding normalized units.

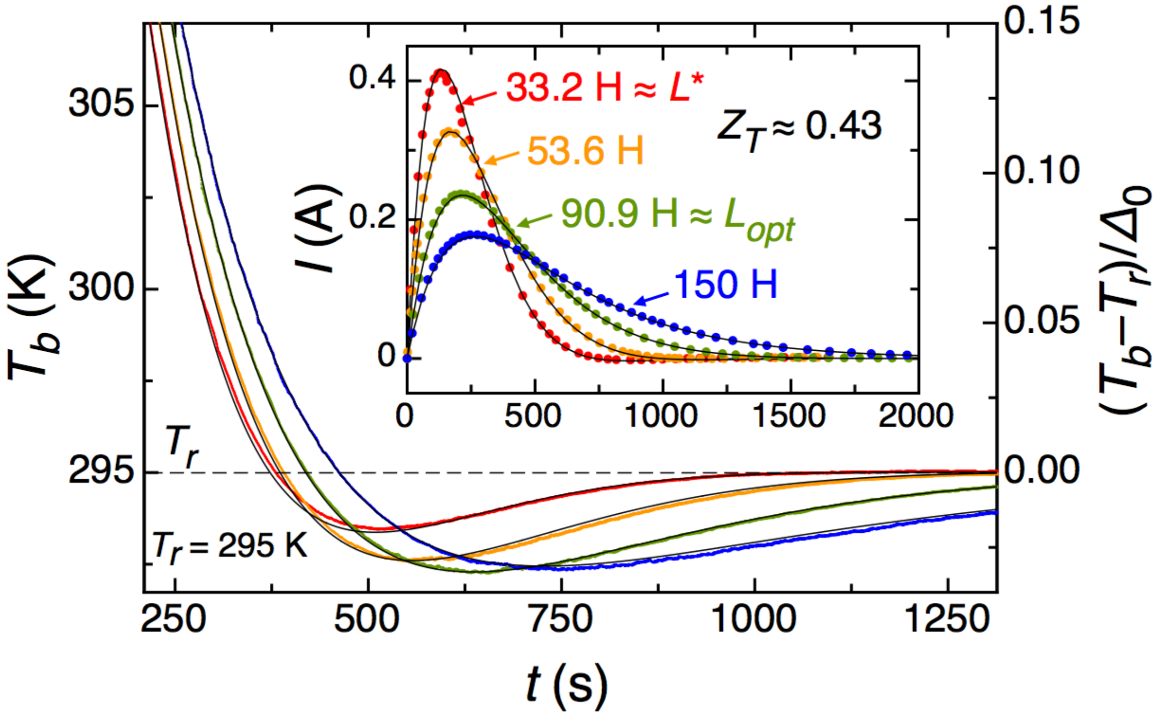

Fig. 4. Results from an experiment with a commercially available Peltier element in combination with a gyrator-type substitute of an electric inductor, representing the equivalent of a "thermal inductor". Main panel: Temperature $T_b(t)$ of a copper cube that was thermally connected to a heat reservoir held at T_r = 295 K and initially heated by $\Delta_0 = T_b(0) - T_r \approx 82\mathrm{K}$, for four different values of nominal inductance L. In combination with the Peltier element, the resulting thermal circuit containing the equivalent of a "thermal inductor" leads to a significant undershoot of $T_b(t)$ with respect to T_r by up to ≈ 2.7 K for L = 90.9 H. Inset: Evolution of the electric current flowing through the Peltier element. The solid black lines correspond to a global fit to all the data according to the relations given in the main text, with the fitting parameters C, R, k and Z_T .

Supplementary Materials

Materials and Methods

We used a commercial Peltier element from *Kryotherm Inc.,* Module TB-7-1.4-2-5, for all the experiments. The temperature difference between the thermal load (a copper cube of dimensions ≈ 10x10x10 mm^3 with a mass $m \approx 9$ g) and the thermal bath (a block of 3.65 kg copper) was measured with a copper-constantan differential thermocouple, thermally connected to the experiment with silver paint. The corresponding voltage was measured using a *Keysight* multimeter 34465 A, and converted according to a standard type T conversion table. The gyrator was built based on the scheme described in [12]. The values for L can be adjusted by an appropriate choice of a resistor in the gyrator circuit, and they have been cross-checked by measuring the time constants of a respective LR test circuit. The current I was determined by monitoring the voltage across an internal shunt resistor inside the gyrator. All experiments were done in open air without any thermal shielding.

In a global fit to all the available $T_b(t)$ and $I(t)$ data we obtain an excellent agreement for C = 4.96 J/K (this value includes a heat-capacity contribution of the Peltier element), R = 0.22 Ω, and effective values for k = 0.0318 W/K and Z_T = 0.432, resulting in L^* = 34.3 H and L_{opt} = 94.4 H. The ratio of the fitted values C and k, $\tau = C/k \approx 156$ s, was independently confirmed by a direct measurement by monitoring the decay of the temperature of the copper cube without the inductor connected and the Peltier circuit open, with $\tau \approx 162$ s. The effective Z_T = 0.432 obtained from the global fitting procedure has to be interpreted as a constant $Z_T = \alpha^2 T_r / kR$, where k and R in the equations (1) may include extrinsic contributions due to parasitic heat transport and resistance from the wiring. The true, intrinsic Z_T value of the Peltier element may therefore be somewhat larger, and we estimate it to $Z_T \approx 0.5$ near room temperature. The fitted value R = 0.22 Ω of the whole circuit including the wiring, is also compatible with the resistance specifications of the Peltier element (R = 0.18 Ω +/– 10%), which represents the main source of electrical resistance.

S1: Maximum possible cooling effect

The maximum possible temperature difference $|T_b - T_r|$ for $t > 0$ is not fully reached for the parameters used in the main text, but with $L_{opt} = \lambda\, L^*$ where $\lambda(Z_T) > 1$. The numerically obtained function $\lambda(Z_T)$ is shown in Fig. S1A. In the limit $Z_T \to \infty$, we have $L_{opt} = L^*$. For a finite Z_T, however, taking $L = L_{opt}$ results in $\omega_{opt} = \omega^* \sqrt{(Z_T+1)/(\lambda Z_T) - (\lambda+1)^2/4\lambda^2}$ and $\tau_{opt} = \tau^* 2\lambda/(1+\lambda)$, and a truly oscillatory behaviour of $I(t)$ is possible only for $Z_T > 1/3$. Even if the solutions for $I(t)$ are over-damped for $Z_T < 1/3$, the temperature of the body still undershoots with the initial conditions used in the main text, and the minimum of $(T_b - T_r)/\Delta_0$ remains negative for all values of Z_T. A choice of $L = L_{opt}$ instead of L^* leads only to a moderately improved cooling performance of the device (Fig. S1B), however, at the cost of significantly increasing the required inductivity value L.

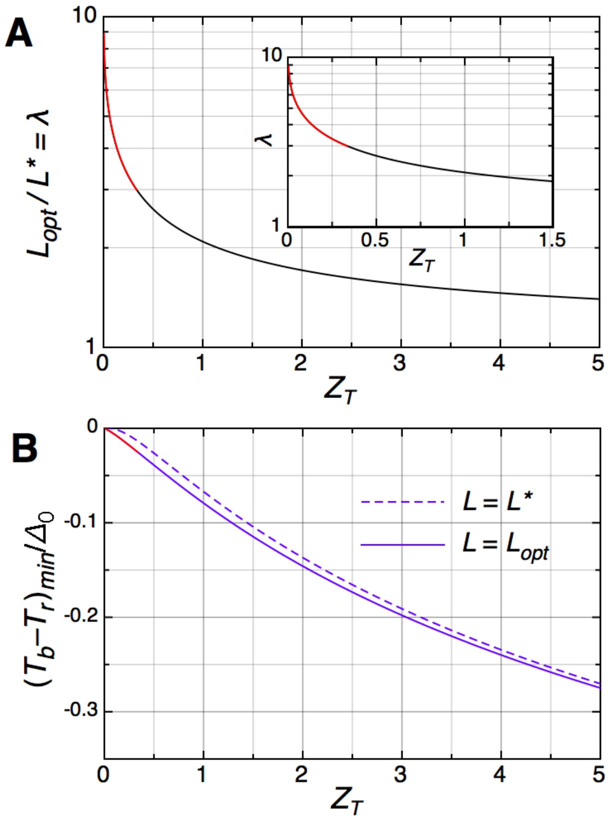

Fig. S1. Optimized values for a maximum temperature undershoot. (**A**) Optimum inductance $L_{opt} = \lambda(Z_T)RC/k$, in units of $L^* = RC/k$. If $L = L_{opt}$, T_b exhibits the maximum possible undershoot with respect to the temperature T_r. (**B**) Minimum value of $(T_b - T_r)/\Delta_0$ relative to the initial temperature difference Δ_0, for $L = L_{opt}$ (solid line) and $L = L^*$ (dashed line). While in the latter case, $I(t)$ shows a true oscillation for all values of Z_T, the current $I(t)$ is over-damped for Z_T < 1/3 when choosing $L = L_{opt}$ but still results in an undershoot of $T_b(t)$ (red lines).

S2: Two finite bodies with different temperatures

If we ascribe a heat capacity $2C$ to each of the bodies, the corresponding amounts of exchanged heat are $\dot{Q}_b = 2C\dot{T}_b$ and $\dot{Q}_r = 2C\dot{T}_r$, respectively. Combining these quantities with the Eqs. 1 in the main text by eliminating both T_b and T_r, we obtain the nonlinear differential equation for $I(t)$

$$LC\ddot{I} + (RC + kL)\dot{I} + (kR + \alpha^2\overline{T})I - \alpha^2 LI^3/8C = 0, \tag{S1}$$

with the mean initial temperature $\bar{T} = [T_b(0) + T_r(0)]/2$. The time dependent temperatures $T_b(t)$ and $T_r(t)$ can again be obtained from the Kirchhoff relation (1a). Additional conditions are the thermal balance equation and the energy conservation law, requiring $2\bar{T} = T_b + T_r + LI^2/4C$ (which can also be derived from the Kirchhoff equation), and $2\bar{T} = T_b + T_r$ for $t \to \infty$ and $I \to 0$ if the two heat capacities are assumed to be temperature independent.

In the limit $\Delta_0 = T_b(0) - T_r(0) << \bar{T}$, the amount of magnetic energy stored in the inductor is vanishingly small, as we have shown in the main text, i.e., $\frac{1}{2}LI^2 << 4C\bar{T}$. Therefore we may neglect the last term in Eq. S1 and end up in this limit with the same differential equation Eq. 3 for $I(t)$ of a damped harmonic oscillator but with T_r replaced by $\bar{T}$.

S3: Analogy to a "thermal inductor"

Thermal circuits can be mapped onto electrical circuits by replacing the thermal conductance with an electrical conductance, heat capacities with electrical capacitances, and temperature differences with voltage differences ΔV. The resulting differential equations for the heat Q and heat currents $I_{th} = \dot{Q}$, or charge and electric currents I, respectively, turn out to be equivalent. A corresponding "thermal inductor" (with thermal self-inductance L_{th}) would have, in a strict sense, to fulfill the proportionality between the time derivative of the thermal current $\dot{I}_{th} = \ddot{Q}$ and the resulting temperature difference ΔT in an analogous way to $L\dot{I} = -\Delta V$, so that $L_{th}\dot{I}_{th} = -\Delta T$.

We consider the case of a finite body in thermal contact with an infinite thermal reservoir as shown in Figs. 1A and 1B in the main text. We again assume that the temperature variations are sufficiently small so that $|T_b(0) - T_r| = \Delta_0 << T_r$, and that we have perfect electrical conductors with R = 0, which could be achieved by the use of superconducting coils and leads. In analogy to an ideal resistanceless electric coil, we also assume perfect thermal insulation between both ends of the "thermal inductor", i.e., with no thermal leakage current in parallel due to a finite thermal conductance k. We then obtain with Eqs. 1a, 1b, k = 0, R = 0 and $\Delta T = T_b - T_r$

$$\dot{I}_{th} = \ddot{Q}_b = -\alpha\dot{T}_b I - \alpha T_b \dot{I} = -\alpha\dot{T}_b I - \alpha^2 T_b \Delta T / L. \qquad \text{(S2)}$$

The analogy $L_{th}\dot{I}_{th} = -\Delta T$ holds if the first summand on the right side of Eq. S2 is negligible, and we then have [1]

$$L_{th} \approx L/(\alpha^2 T_b) \approx L/(\alpha^2 T_r) = L/(Z_T k R). \qquad \text{(S3)}$$

To fulfill this condition for an arbitrary variation of the electric current with time, $I(t) = I_0 f(t)$, we must require $|\dot{T}_b / T_b| << |\dot{I} / I|$, or in terms of $f(t)$,

$$\frac{\Delta_0}{T_r}\left|\frac{\ddot{f}(t)}{\dot{f}(0)}\right| << \left|\frac{\dot{f}(t)}{f(t)}\right|. \tag{S4}$$

For $\Delta_0 << T_r$ this is fulfilled all the time in most conceivable cases where the variations in $T_b(t)$ are of the order of $\Delta_0 << T_r$, e.g., for $f(t) = \gamma t$, $f(t) = \exp(-t/\tau)$, or $f(t) = 1 - \exp(-t/\tau)$. In this sense, it is justified to interpret the present circuit as the equivalent of a "thermal inductor". For oscillatory functions where $\dot{f}(t)$ temporarily vanishes while $\ddot{f}(t)$ remains finite as in our case of a damped oscillation of $f(t)$, the condition (S4) is momentarily violated, but it is still respected most of the time as a result of the exponential decay of the oscillating electric current, and $\Delta_0 << T_r$.

In this sense, we can re-write Eq. (3) in the main text in terms of only thermal quantities,

$$L_{th} C \ddot{I}_{th} + \left(\frac{C}{Z_T k} + k L_{th}\right)\dot{I}_{th} + \left(\frac{1}{Z_T} + 1\right) I_{th} \approx 0 . \tag{S5}$$

Oscillatory solutions for the thermal current $I_{th}(t)$ are possible for all values of $Z_T > 0$ if $L_{th} = C / (Z_T k^2)$ where $L = L^* = RC / k$.